JHEP HYPER VERSION

Received: *April 13, 2000*, Accepted: *June 4, 2000*

Invasion of the giant gravitons from anti-de Sitter space

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Abstract: It has been known for some time that the AdS/CFT correspondence predicts a limit on the number of single particle states propagating on the compact spherical component of the $AdS \times S$ geometry. The limit is called the stringy exclusion principle. The physical origin of this effect has been obscure but it is usually thought of as a feature of very small distance physics. In this paper we will show that the stringy exclusion principle is due to a surprising large distance phenomenon. The massless single particle states become progressively less and less point-like as their angular momentum increases. In fact they blow up into spherical branes of increasing size. The exclusion principle is simply understood as the condition that the particle should not be bigger than the sphere that contains it.

Keywords: Superstrings and Heterotic Strings, M-Theory, D-branes.

JHEP06(2000)008

Contents

1. Introduction

In conventional (20th century) physics, high energy or high momentum came to be associated with small distances. The physics of the 21st century is likely to be dominated by a very different perspective. According to the Infrared/Ultraviolet connection [1] which underlies much of our new understanding of string theory and its connection to gravity, physics of increasing energy or momentum is governed by increasingly large distances. Examples include the growth of particle size with momentum [2, 3], the origin of the long-range gravitational force in Matrix Theory from high energy quantum corrections [4] and the IR/UV connection in AdS spaces. Another important manifestation is the spacetime uncertainty principle of string theory [5, 6, 7]

$$\Delta x \, \Delta t \sim \alpha' \, . \tag{1.1}$$

Similar uncertainty principles occur in non-commutative geometry where the coordinates of space do not commute. An important consequence of the non-commutativity is the fact that the particles described by non-commutative field theories have a spatial extension which is proportional to their momentum [8, 9]. This in turn leads to unfamiliar violations of the conventional decoupling of IR and UV degrees of freedom in these theories [10, 11, 12]. In this paper we will describe another example of IR/UV non decoupling that occurs in AdS/CFT theories. The relevant space-times have the form $AdS_n \times S^m$. We are interested in the motion of the graviton and other massless bulk particles on the S^m. The motion is characterized by an angular momentum L

or more exactly a representation of the rotation group $O(m+1)$. In 20th century physics such particles are regarded as point or almost point particles regardless of L. In fact we will see that as L increases the particles blow up in size very much like the quanta of non-commutative field theories. When the size reaches the radius of the S^m, the growth can no longer continue and the tower of Kaluza-Klein states terminates. This is the origin of the stringy exclusion principle [13, 14, 15].

In section 2 we will review the theory of electric dipoles moving in a magnetic field. This system is the basic object of non-commutative field theory. When the theory is defined on a 2-sphere there is a bound on the angular momentum when the ends of the dipole separate to the antipodes of the sphere. In sections 3.1, 3.2 and 3.3 we consider the cases of $AdS_7 \times S^4$, $AdS_5 \times S^5$ and $AdS_4 \times S^7$. In each case we find that the spectrum of angular momentum is bounded and that the bound agrees with expectations from the stringy exclusion principle.

2. Dipoles in magnetic fields

In this section we briefly review the dipole analogy for non-commutative field theory [8, 9]. We begin with a pair of unit charges of opposite sign moving on a plane with a constant magnetic field B. The lagrangian is

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) + \frac{B}{2}\,\epsilon_{ij}\left(\dot{x}_1^i\, x_1^j - \dot{x}_2^i\, x_2^j\right) - \frac{K}{2}\left(x_1 - x_2\right)^2 . \tag{2.1}$$

Let us suppose that the mass is so small so that the first term in eq. (2.1) can be ignored. Let us also introduce center of mass and relative coordinates

$$X = \frac{x_1 + x_2}{2}\,, \qquad \Delta = \frac{x_1 - x_2}{2}\,. \tag{2.2}$$

The lagrangian becomes

$$\mathcal{L} = B\,\epsilon_{ij}\,\dot{X}^i\,\Delta^j - 2\,K\,\Delta^2\,. \tag{2.3}$$

From eq. (2.3) we first of all see that X and Δ are non-commuting variables satisfying

$$\left[X^i, \Delta^j\right] = i\,\frac{\epsilon^{ij}}{B}\,. \tag{2.4}$$

Furthermore the center of mass momentum conjugate to X is

$$P_i = B\,\epsilon_{ij}\,\Delta^j\,. \tag{2.5}$$

Thus when the dipole is moving with momentum P in some direction it is stretched to a size

$$|\Delta| = \frac{|P|}{B} \tag{2.6}$$

in the perpendicular direction. This is the basis for the peculiar non-local effects in non-commutative field theory.

JHEP06(2000)008

Now suppose the dipole is moving on the surface of a sphere of radius R. Assume also that the sphere has a magnetic flux N. In other words there is a magnetic monopole of strength

$$2\,\pi N = \Omega_2\, B\, R^2 \tag{2.7}$$

at the center of the sphere. We can get a rough idea of what happens by just saying that when the momentum of the dipole is about $2BR$ the dipole will be as big as the sphere. At this point the angular momentum is the maximum value

$$L = PR \sim BR^2\,. \tag{2.8}$$

This is of order the total magnetic flux N.

We will now do a more precise analysis and see that the maximum angular momentum is exactly N. Parameterize the sphere by two angles θ, ϕ. The angle ϕ measures angular distance from the equator. It is $\pm\pi/2$ at the poles. The azimuthal angle θ goes from 0 to 2π. We work in a gauge in which the θ component of the vector potential is non zero. It is given by

$$A_\theta = N\,\frac{1-\sin\phi}{2\,R\cos\phi}\,. \tag{2.9}$$

For a unit charged point particle moving on the sphere the term coupling the velocity to the vector potential is

$$\mathcal{L}_A = A_\theta\, R\cos\phi\,\dot\theta = NR\,\frac{1-\sin\phi}{2R}\,\dot\theta\,. \tag{2.10}$$

Now consider a dipole with its center of mass moving on the equator. The positive charge is at position (θ, ϕ) and the negative charge is at $(\theta, -\phi)$. For the motion we consider ϕ is time independent. Equation (2.10) becomes

$$\mathcal{L}_A = N\left(\frac{1-\sin\phi}{2}\right)\dot\theta - N\left(\frac{1+\sin\phi}{2}\right)\dot\theta \tag{2.11}$$

or

$$\mathcal{L}_A = -N\sin\phi\,\dot\theta\,. \tag{2.12}$$

Again we want to consider a slow-moving dipole whose mass is so small that its kinetic term may be ignored compared to the coupling to the magnetic field, i.e. $mR \ll N$. Let us also add a spring coupling

$$\mathcal{L}_S = -\frac{k}{2}\,R^2\sin^2\phi\,; \tag{2.13}$$

for simplicity, we have used the chordal distance in this potential. The total lagrangian is

$$\mathcal{L} = -\frac{k}{2}\,R^2\sin^2\phi - N\sin\phi\,\dot\theta \tag{2.14}$$

and the angular momentum is

$$L = -N \sin\phi \,. \tag{2.15}$$

The angular momentum will reach its maximum when $\phi = \pi/2$ at which point

$$|L_{\max}| = N \,. \tag{2.16}$$

The fact that the angular momentum of a single field quantum in non-commutative field theory is bounded by N is well known in the context of non-commutative field theory on a sphere [20]. Here we see that it is a large distance effect.

3. $AdS \times S$

We now study BPS particles moving on the sphere of maximally supersymmetric AdS vacua of string and M-theory. For simplicity, we present all details of the argument in the case of the M5-brane geometry. Results for the other cases are given in the latter subsections. Note that in all of these cases, the energy of our objects is well below the energy of a stable AdS black hole.

3.1 $AdS_7 \times S^4$

We are interested in the motion of a BPS particle on the 4-sphere of $AdS_7 \times S^4$. We will assume that the radius of curvature R is much larger than the 11-dimensional Planck length l_p. The analogy with the previous example is very close. The role of the magnetic field is played by the 4-form field strength on the sphere. We call the flux density B. Quantization of flux requires

$$\Omega_4 B R^4 = 2\,\pi N \,. \tag{3.1}$$

From the supergravity equations of motion it can be seen [17] that R is given by

$$R = l_p\,(\pi N)^{1/3} \,. \tag{3.2}$$

The assumption of large R means $N \gg 1$.

We want to know what happens to a graviton or any other massless 11-dimensional particle when it moves on the 4-sphere in the presence of the 4-form field strength. As long as the angular momentum is small ($L \ll N$) the graviton is expected to be much smaller than R, and we can make the approximation that space is locally flat. Furthermore from eq. (3.1) we see that the field strength B is small

$$B \sim N^{-1/3}\, l_p^{-4} \,. \tag{3.3}$$

Since the space is almost flat we can locally introduce flat space coordinates $x^0, \ldots, x^{10}$. Let us take the B-field to lie in the $(7,8,9,10)$ directions. The graviton is moving along the x^{10} direction. Its momentum is $P_{10} = L/R$. This setup is very close to one that was studied by Myers using Matrix Theory [16].

In matrix theory the 11-dimensional graviton is viewed as a threshold bound state of $n = P_{10}R_{10}$ D0-branes. Myers shows that in a background 4-form fieldstrength the D0-brane configuration is described as a spherical membrane with a radius r that grows with P_{10} according to

$$r \sim BP_{10}\, l_p^6\,. \tag{3.4}$$

Let us assume that this formula is approximately valid until r becomes of order R. In that case when the graviton size becomes $\sim R$ it will have momentum

$$P_{10}(\max) \sim \frac{R}{B\, l_p^6} \tag{3.5}$$

and angular momentum

$$L_{\max} \sim R\, P_{10}(\max) \sim \frac{R^2}{B\, l_p^6} \sim N\,. \tag{3.6}$$

Thus as in the previous section we find that the maximum single particle angular momentum is N.

We will now give a more precise calculation which parallels that of section 1. We are interested in the dynamics of a relativistic spherical membrane moving in S^4. The membrane has zero net charge but it couples to the background field strength. It behaves like the dipole of section 1.

Let us parametrize S^4 using cartesian coordinates $X_1, \ldots, X_5$ so that

$$\begin{gathered} X_1 = R\,\cos\theta_1\,, \qquad X_2 = R\,\sin\theta_1\cos\theta_2\,, \qquad X_3 = R\,\sin\theta_1\sin\theta_2\cos\theta_3\,, \\ X_4 = R\,\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4\,, \qquad X_5 = R\,\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\,. \end{gathered} \tag{3.7}$$

The angles $\theta_1, \ldots, \theta_3$ go from 0 to π. The angle θ_4 is the azimuthal angle and goes from 0 to 2π. Then

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 = R^2\,. \tag{3.8}$$

Next we embed a spherical membrane in S^4. We choose to parametrize the surface of the membrane by θ_3, θ_4. The brane is allowed to move in the X_1, X_2 plane. Its size depends on its location in the X_1, X_2 plane according to

$$r = R\,\sin\theta_1\sin\theta_2\,. \tag{3.9}$$

We see that when the size is at its maximum value, $r = R$, the membrane is at the origin $X_1 = X_2 = 0$ like the two charges at the ends of the dipole in section 1. Since

$$X_1^2 + X_2^2 = R^2 - r^2\,, \tag{3.10}$$

the membrane can move around a circle in the plane and have constant size. We also set

$$X_1 = \sqrt{R^2 - r^2}\cos\phi\,, \qquad X_2 = \sqrt{R^2 - r^2}\sin\phi\,. \tag{3.11}$$

In terms of the coordinates $r, \phi, \theta_3, \theta_4$, the metric on the 4-sphere becomes

$$ds^2 = \frac{R^2}{(R^2 - r^2)}\, dr^2 + \left(R^2 - r^2\right) d\phi^2 + r^2\, d\Omega_2^2\,, \tag{3.12}$$

where $d\Omega_2^2$ is the metric of a unit 2-sphere parametrized by θ_3 and θ_4. From the metric we see that the volume element is just

$$R\, r^2\, dr\, d\phi\, d\Omega_2\,. \tag{3.13}$$

The kinetic energy of the membrane is given by the Dirac-Born-Infeld lagrangian. We are mostly interested in the case when the size of the sphere r is constant and close to its maximum value R. In this case the membrane moves around a circle in the X_1, X_2 plane of radius $(R^2 - r^2)^{1/2}$ with angular velocity $\dot{\phi}$. Dropping time derivatives of r, we have

$$\mathcal{L}_K = -T\, \Omega_2\, r^2 \sqrt{1 - (R^2 - r^2)\, \dot{\phi}^2}\,. \tag{3.14}$$

Here, T is the membrane tension which is given by

$$T = \frac{1}{4\, \pi^2\, l_p^3} \tag{3.15}$$

in 11-dimensional Planck units.

Next we add the Chern-Simons coupling involving the background field. The contribution of the 4-form field strength to the action of the brane per orbit around the S^4 is

$$S_B = \int_{wv} C = \int_{\Sigma} F\,. \tag{3.16}$$

The first integral is over the world-volume of the brane. $F = dC$ is the 4-form flux, and Σ is a four-manifold in the S^4 whose boundary is the 3-dimensional surface swept out by the brane during one orbit. Since the background flux is just $F = B d\mathrm{vol}$, where B is the constant flux density and $d\mathrm{vol}$ is the volume form on S^4, we have

$$S_B = B\, \mathrm{vol}(\Sigma)\,. \tag{3.17}$$

Therefore the Chern-Simons term in the lagrangian is

$$\mathcal{L}_B = \frac{S_B}{T} = B\, \mathrm{vol}(\Sigma)\, \frac{\dot{\phi}}{2\pi}\,, \tag{3.18}$$

where $\dot{\phi}$ is the (constant) angular velocity of the brane. Parametrizing the motion as above, the volume of Σ is

$$\mathrm{vol}(\Sigma) = R\, \Omega_2 \int_0^{2\pi} d\phi \int_0^r r'^2\, dr' = \frac{8\, \pi^2}{3}\, R\, r^3\,. \tag{3.19}$$

So the Chern-Simons term is

$$\mathcal{L}_B = \frac{\dot\phi}{2\pi}\, B\,\Omega_4\, R\, r^3 = \dot\phi\, N\,\frac{r^3}{R^3}\,, \tag{3.20}$$

where we used the flux quantization condition, eq. (3.1).

Therefore, the full bosonic lagrangian is

$$\mathcal{L} = -m\sqrt{1-\dot\phi^2\,(R^2-r^2)} + N\,\frac{r^3}{R^3}\,\dot\phi \tag{3.21}$$

with $m = \Omega_2 T r^2$. Using eqs. (3.2) and (3.15), we also see that

$$\frac{N}{R^3} = T\,\Omega_2\,. \tag{3.22}$$

From the lagrangian we find that the angular momentum is given by

$$L = \frac{m\,\dot\phi\,(R^2-r^2)}{\sqrt{1-\dot\phi^2\,(R^2-r^2)}} + N\,\frac{r^3}{R^3}\,. \tag{3.23}$$

The maximum size a membrane can have is R. Also, the velocity of its center of mass, $\dot\phi R$, cannot exceed the speed of light. This implies that the angular momentum has a maximum value given by N:[1]

$$L_{\max} = N\,. \tag{3.24}$$

When the membrane has maximal size R, the angular momentum is the maximum value N. We see that the Kaluza-Klein graviton has a maximum angular momentum in agreement with the stringy exclusion principle. For the energy, we find

$$E = \dot\phi\, L - \mathcal{L} = \sqrt{\left(\frac{Nr^2}{R^3}\right)^2 + \frac{(L-Nr^3/R^3)^2}{R^2-r^2}}\,. \tag{3.25}$$

Varying the energy with respect to r at fixed L, we find

$$\frac{dE}{dr} = \frac{r}{E\,(R^2-r^2)^2}\left(L - N\frac{r}{R}\right)\left(L - 2N\,\frac{r}{R} + N\,\frac{r^3}{R^3}\right). \tag{3.26}$$

We see that for $L < N$ there exists a stable minimum at

$$r = \frac{L}{N}\,R\,. \tag{3.27}$$

Therefore, the membrane grows as we increase the angular momentum. This is in agreement with eq. (3.4).[2] When $r = R$ and $L = N$, a more careful analysis for the

[1] There is an exception to this statement at the pathological value $r = 0$ which we discuss at the end of this section.

[2] The cubic factor in dE/dr has a positive root at $0 < r < (L/N)R$ at which the energy has a local maximum and another root at $r > R$.

stability of the solution is needed and we do so at the end of this section. The value of the energy at the minimum is

$$E = \frac{L}{R}\,, \tag{3.28}$$

which is the energy of a Kaluza-Klein graviton with angular momentum L. From eq. (3.23), we also find that the velocity of the center of mass equals the speed of light, $\dot{\phi} R = 1$.

We now show that there is a stable solution at $r = R$ and $L = N$. Setting $\tilde{r} = R - r$ and expanding the lagrangian up to quadratic powers in $\tilde{r}$, we obtain

$$\mathcal{L}_K = -T\,\Omega_2\,R^2 + T\,\Omega_2\,R\tilde{r}\left(2 + \dot{\phi}^2\,R^2\right) - T\,\Omega_2\,\tilde{r}^2\left(\dot{\phi}^4\,R^4 + \frac{5}{2}\,\dot{\phi}^2\,R^2 + 1\right), \tag{3.29}$$

and

$$\mathcal{L}_B = \Omega_4\,B\,R^4\,\dot{\phi}\left(1 - 3\frac{\tilde{r}}{R} + 3\,\frac{\tilde{r}^2}{R^2}\right) = N\,\dot{\phi} - R\,\tilde{r}\,\frac{3N}{R^3}\,\dot{\phi}\,R + \tilde{r}^2\,\frac{3N}{R^3}\,\dot{\phi}R\,. \tag{3.30}$$

Using $N/R^3 = T\Omega_2$, the total lagrangian becomes

$$\begin{aligned}\mathcal{L} = {} & -T\,\Omega_2\,R^2 + N\,\dot{\phi} + T\,\Omega_2\,R\,\tilde{r}\left(2 - 3\,\dot{\phi}R + \dot{\phi}^2\,R^2\right) - \\ & - T\,\Omega_2\,\tilde{r}^2\left(\dot{\phi}^4\,R^4 + \frac{5}{2}\,\dot{\phi}^2\,R^2 - 3\,\dot{\phi}\,R + 1\right).\end{aligned} \tag{3.31}$$

There is an extremum at $\tilde{r} = 0$ provided that

$$2 - 3\,\dot{\phi}\,R + \dot{\phi}^2\,R^2 = 0\,. \tag{3.32}$$

This can be achieved if the velocity $\dot{\phi}R = 1$. Thus, when the size of the membrane is R its center of mass moves with the speed of light. Furthermore, the extremum is stable since

$$\left.\frac{d^2V(r)}{d^2r}\right|_{r=R} > 0\,. \tag{3.33}$$

Thus when the size of the membrane is R, the angular momentum has its maximum value N and the energy is given by

$$E = T\,\Omega_2\,R^2 = \frac{N}{R}\,. \tag{3.34}$$

At the classical level, this is in exact agreement with the energy of a Kaluza-Klein graviton having angular momentum N about the sphere. When $N \gg 1$, the maximal angular momentum is large and the classical formula for the energy agrees with the BPS bound for the energy given the angular momentum. Quantum corrections are suppressed by $1/N$. The Kaluza-Klein graviton is a BPS state and its energy should not change under the process of blowing up into a membrane. Again, the size of the brane is determined by the angular momentum. Since the maximum size a brane can have is R, there is a maximum angular momentum as predicted by the dual conformal field theory. The fact that the energy of the brane agrees with the BPS formula for given angular momentum is a non-trivial test of our model.

We also note that there is a minimum of the energy at $r = 0$ as well. Classically, it corresponds to a massless particle moving around the equator with angular momentum L.[3] Such a solution is singular from the perspective of the gravitational field equations since for angular momenta of order N, it represents a huge energy (of order $N^{2/3}$) concentrated at a point. Therefore it is subject to uncontrolled quantum corrections. In particular, there are quantum corrections proportional to powers of the momentum times the flux density, which are large at angular momenta of order N.

We have shown in this paper that such a singular solution can be resolved by blowing into a smooth macroscopic membrane of size $(L/N)R$. Our classical analysis is expected to be valid for the large membrane. The smooth membrane solution certainly has much more nearby phase space and as a result the true quantum ground state will be overwhelmingly supported at the membrane solution. We believe that this is another example of how string/M-theory resolves singularities of the type studied recently in ref. [21].

3.2 $AdS_5 \times S^5$

The extension of our analysis to the other two maximally supersymmetric cases is straightforward and we will be much less explicit. Consider the case of $AdS_5 \times S^5$ first. The radius of the 5-sphere is given by

$$R = (4\pi\, g_s\, N)^{1/4}\, l_s\,, \tag{3.35}$$

where g_s, l_s are the string coupling constant and string length scale and N is the number of units of 5-form flux on the sphere. We take N large keeping $g_s N$ fixed and large.

In type-IIB string theory on $AdS_5 \times S^5$ with $N \gg 1$, the maximum angular momentum of a BPS particle on the S^5 is N [13]. From the gauge theory perspective, this can be seen from the fact that one builds up such states by a single trace of the $N \times N$ scalars in the $\mathbf{6}$ of SO(6). The largest representation of SO(6) one can build in this way is the spin-N representation, $\mathrm{Sym}^N \mathbf{6}$.

From our perspective this is because the particle moving on the sphere expands into a spherical D3-brane. In this case we present only the exact classical analysis of the D3-brane wrapping an S^3 that moves in S^5. The bosonic lagrangian is

$$\mathcal{L} = \mathcal{L}_{DBI} + \mathcal{L}_{CS} = -T_{D3}\, \Omega_3\, r^3 \sqrt{1 - (R^2 - r^2)\, \dot{\phi}^2} + \dot{\phi}\, N\, \frac{r^4}{R^4}\,. \tag{3.36}$$

[3]We thank Sunny Itzhaki for discussions of this point.

The tension of the D3-brane is

$$T_{D3} = \frac{1}{(2\pi)^3\, l_s^4\, g_s}\,. \tag{3.37}$$

We will use the relation

$$T_{D3}\,\Omega_3 = \frac{N}{R^4}\,. \tag{3.38}$$

The angular momentum in terms of $\dot\phi$ is

$$L = \frac{m\,\dot\phi\,(R^2 - r^2)}{\sqrt{1 - \dot\phi^2\,(R^2 - r^2)}} + N\,\frac{r^4}{R^4}\,, \tag{3.39}$$

where $m = T_{D3}\Omega_3 r^3 = (N/R^4) r^3$. Again we see that the angular momentum is bounded by N since $0 \le r \le R$ and $0 \le \dot\phi R \le 1$. The energy is

$$E = \sqrt{m^2 + \frac{\left(L - N r^4/R^4\right)^2}{R^2 - r^2}}\,. \tag{3.40}$$

Varying the energy with respect to r at fixed L, we find in this case a stable minimum when

$$r^2 = \frac{L}{N}\,R^2\,. \tag{3.41}$$

The value of the energy at this minimum again matches the BPS bound when L is large, for $N \gg 1$:

$$E = \frac{L}{R}\,. \tag{3.42}$$

This is strong evidence that at any appreciable momentum, at least at the (semi-) classical level, the good description of Kaluza-Klein gravitons is in terms of branes, rather than fundamental strings. From the dual CFT, we know that there is a unique BPS state with these quantum numbers; consistency with the exclusion principle implies that it is the one described by the spherical brane.

3.3 $AdS_4 \times S^7$

In this case, we expect the graviton to expand into an M5-brane which is an $S^5 \subset S^7$. The radius of the sphere is given by

$$R = \left(2^5\,\pi^2\,N\right)^{1/6} l_p\,. \tag{3.43}$$

The tension of the 5-brane is given by

$$T = \frac{1}{(2\pi)^5\, l_p^6} \tag{3.44}$$

and we have the relation

$$m = T\,\Omega_5\, r^5 = \frac{N}{R^6}\, r^5\,. \tag{3.45}$$

The lagrangian is

$$\mathcal{L} = -T\,\Omega_5\,r^5\,\sqrt{1-(R^2-r^2)\,\dot\phi^2} + \dot\phi\,N\,\frac{r^6}{R^6}\,. \tag{3.46}$$

The angular momentum in terms of $\dot\phi$ is

$$L = \frac{m\,\dot\phi\,(R^2-r^2)}{\sqrt{1-\dot\phi^2\,(R^2-r^2)}} + N\,\frac{r^6}{R^6}\,. \tag{3.47}$$

The energy is

$$E = \sqrt{m^2 + \frac{\left(L - N\,r^6/R^6\right)^2}{R^2-r^2}}\,. \tag{3.48}$$

Varying the energy with respect to r at fixed L, we find in this case a stable minimum when

$$r^4 = \frac{L}{N}\,R^4\,. \tag{3.49}$$

The value of the energy at this minimum again matches the BPS bound when L is large:

$$E = \frac{L}{R}\,. \tag{3.50}$$

3.4 Remarks about AdS_3

We will conclude this section with some comments about $AdS_3 \times S^3 \times M_4$. This case is distinguished in several ways.

First, it is not clear into what the graviton should expand. Consider the geometry built from the D1-D5 system with Q_1 D-strings and Q_5 5-branes. The stringy exclusion bound on the angular momentum is $L \leq Q_1 Q_5$. The graviton is expected to blow up into a circular string moving on the S^3, but should it be a D5-brane wrapped on the four-manifold, or a D-string?

Secondly, the energetic considerations degenerate in this case. If we assume for argument's sake that the graviton blows up into either a D-string or wrapped 5-brane which is a circle on the S^3, we find that the energy at fixed L has no nontrivial minimum. Considering some incarnation of the "fractional strings" of [18] does not help.

It may help to consider the S-dual situation of the F1-NS5 system. In this case, the dynamics of fundamental strings on the relevant sphere are described by a level-Q_5 SU(2) WZW model. One expects the exclusion principle to be related to the affine cutoff on SU(2) representations. Finally, a clarification of this case should match the result of [19] that the exclusion bound occurs at the critical value of the energy for black hole formation. A better understanding remains for future work.

4. Conclusions

Physics in non-commutative spaces is characterized by a simple signature — the increase of size on systems with increasing momentum. In this paper we have seen that the motion of massless quanta on the S factor of $AdS_n \times S^m$ has exactly this behavior. The massless particle blows up into a spherical brane of dimensionality $m-2$ whose radius increases with increasing momentum. Eventually the radius of the blown-up brane becomes equal to the radius of the sphere that contains it. It can no longer grow and the spectrum is terminated. This is the origin of the stringy exclusion principle. Thus we see one more piece of evidence for non-commutativity of space in quantum gravity.

Acknowledgments

We would like to thank Allan Adams, Micha Berkooz, Raphael Bousso, Sumit Das, Antal Jevicki, Jason Prince Kumar, Albion Lawrence, Alec Matusis and Joe Polchinski for conversations. We are especially grateful to Joe Polchinski for explaining to us the flat-space energetics. We thank Hirosi Ooguri and Sunny Itzhaki for comments about the first version. We would also like to thank Steve Shenker for help with the title. The work of JM is supported in part by the Department of Defense NDSEG fellowship program. The work of LS and NT is supported in part by NSF grant 9870115.

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